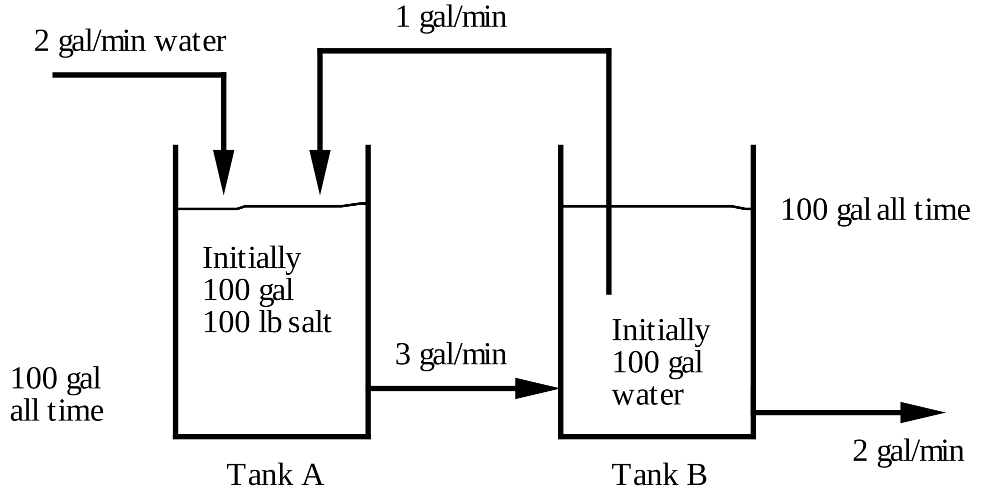
SYSTEMS OF DIFFERENTIAL EQUATIONS

1 Introduction - An Example

Let tank A contain 100 gallons of brine in which 100 lbs of salt is dissolved and tank B contain 100 gallons of water. Suppose water flows into tank A at the rate of 2 gallons per minute, and the mixture flows from tank A into tank B at 3 gallons per minute. From B, 1 gallon per minute is pumped back to A while 2 gallons per minute are flushed away. We wish to find the amount of salt in both tanks at all time t.[1]



Assume:

$x(t)$ = number of pounds of salt in tank A at time t

$y(t)$ = number of pounds of salt in tank B at time t

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1 From "Advanced Engineering Mathematics", by Grossman, S. I., and Derrick, W. R., p. 128, 1988.

Since from material balance

{Rate of change in x} = { Input rate} – { Output rate}

and the volume s of water in both tanks are constants, we have

Tank A: $$\frac{dx}{dt} = \frac{y}{100}\times 1 - \frac{x}{100}\times 3 = -3\frac{x}{100} + \frac{y}{100} \qquad (1)$$

Tank B: $$\frac{dy}{dt} = \frac{x}{100}\times 3 - \frac{y}{100}\times 1 - \frac{y}{100}\times 2 = 3\times\frac{x}{100} - 3\times\frac{y}{100} \qquad (2)$$

initial conditions:

$$x(t=0) = 100, \qquad y(t=0) = 0$$

Eliminate one of the dependent variables, say x, by

Since from (2)

$$y' = \frac{3}{100}x - \frac{3}{100}y$$

$$\therefore \qquad x = y + \frac{100}{3}y' \qquad (3)$$

$$\Rightarrow \qquad x' = y' + \frac{100}{3}y'' \qquad (4)$$

Put (3) and (4) into (1), we have

$$y' + \frac{100}{3}y'' = \frac{-3}{100}\left[y + \frac{100}{3}y'\right] + \frac{1}{100}y \quad \text{or} \quad \frac{100}{3}y'' + 2y' + \frac{2}{100}y = 0 \qquad (5)$$

Since the above equation is a second–order, constant-coefficient, linear differential equation in y, we need 2 initial conditions for y:

$$y(0) = 0, \text{ but } y'(0) = ?$$

$$\text{From (2)} \qquad y'(0) = \frac{3}{100}(x(0) - y(0)) = \frac{3}{100}(100 - 0) = 3$$

$\therefore$ Equation (5) becomes

$$\mathbf{y'' + \frac{6}{100} y' + \frac{6}{(100)^2} y = 0;}$$

$$y(0) = 0, \quad y'(0) = 3$$

The characteristic equation of the above differential equation is

$$\lambda^2 + \frac{6}{100}\lambda + \frac{6}{(100)^2} = 0 \quad \text{or} \quad \lambda = \frac{-3 \pm \sqrt{3}}{100}$$

thus $$y(t) = C_1 \exp\left\{\frac{-3+\sqrt{3}}{100} t\right\} + C_2 \exp\left\{\frac{-3-\sqrt{3}}{100} t\right\}$$

Put $y(0) = 0$ and $y'(0) = 3$ into the above equation, we have $C_1 = -C_2 = 50\sqrt{3}$

$$\Rightarrow \qquad y(t) = 50\sqrt{3}\left\{ e^{\frac{-3+\sqrt{3}}{100}t} - e^{\frac{-3-\sqrt{3}}{100}t} \right\}$$

From Equation (3), we have

$$x(t) = y + \frac{100}{3} y' = 50 \left\{ e^{\frac{-3+\sqrt{3}}{100} t} + e^{\frac{-3-\sqrt{3}}{100} t} \right\}$$

In general, we have three methods to solve systems of differential equations:

(1) **Method of Elimination** – Present Example
(2) **Method of Determinants** – Differential Operator
(3) **Matrix Method** – Will be discussed in the Chapter - Matrix

2 Method of Elimination

$$x' = a_1 x + b_1 y + f_1(t) \tag{1}$$
$$y' = a_2 x + b_2 y + f_2(t) \tag{2}$$

where x' and y' are dx/dt and dy/dt, respectively. If

$$f_1(t) = f_2(t) = 0 \quad \Rightarrow \quad \text{Homogeneous}$$
$$f_1(t) \neq 0 \ \text{ or } \ f_2(t) \neq 0 \quad \Rightarrow \quad \text{Nonhomogeneous}$$

In general, the above equations can be solved by writing equation (1) into

$$b_1 y = x' - a_1 x - f_1(t) \tag{3}$$

Differentiating (1) gives

$$x'' = a_1 x' + b_1 y' + f_1'(t) \tag{4}$$

Put eqns. (2) and (3) into the above equation, we have

$$\begin{aligned} x'' &= a_1 x' + b_1 [a_2 x + b_2 y + f_2(t)] + f_1' \\ &= a_1 x' + b_1 a_2 x + b_2 [x' - a_1 x - f_1(t)] + b_1 f_2 + f_1' \end{aligned}$$

$$\Rightarrow \qquad x'' - (a_1 + b_2) x' + (a_1 b_2 - b_1 a_2) x = b_1 f_2 - b_2 f_1 + f_1'$$

which is a nonhomogeneous linear second–order differential equation of x.

[Example]

$$x' = 2x + y + t \qquad (1)$$

$$y' = x + 2y + t^2 \qquad (2)$$

[Solution] From (1), we have $y = x' - 2x - t$ and

$$x'' = 2x' + y' + 1 = 2x' + (x + 2y + t^2) + 1 = 2x' + x + 2(x' - 2x - t) + t^2 + 1$$

$$\Rightarrow \qquad x'' - 4x' + 3x = (t-1)^2$$

$$\Rightarrow \qquad x(t) = c_1 e^t + c_2 e^{3t} + \frac{1}{3}t^2 + \frac{2}{9}t + \frac{11}{27}$$

$$y(t) = -c_1 e^t + c_2 e^{3t} - \frac{2}{3}t^2 - \frac{7}{9}t - \frac{16}{27}$$

[Exercise]

$$x' = -2x + y$$

$$y' = -4x + 3y + 10\cos t$$

[Exercise]

$$Dx + y = \cos t - \sin t$$

$$Dy + x = \cos t + \sin t \qquad \text{where } D = d/dt$$

[Exercise]

$$x' = \frac{3}{t}x + \frac{1}{t}y$$

$$y' = -\frac{4}{t}x - \frac{1}{t}y$$

[Ans.]

$$x = \left[-\frac{c_1}{2} - \frac{c_2}{4}\right]t - \frac{c_2}{2}t\ln t$$

$$y = c_1 t + c_2 t\ln t$$

3 Method of Determinants

Recall that the differential equation

$$y'' + y' - 6y = 0$$

can be written in terms of differential operator D:

$$(D^2 + D - 6)\, y = 0$$

Thus, for a system of two second–order equations

$$\begin{aligned} a_{11} x'' + b_{11} x' + c_{11} x + a_{12} y'' + b_{12} y' + c_{12} y &= \phi_1(t) \\ a_{21} x'' + b_{21} x' + c_{21} x + a_{22} y'' + b_{22} y' + c_{22} y &= \phi_2(t) \end{aligned}$$

where a_{ij}, b_{ij} and c_{ij} are constants, can be written as

$$\begin{aligned} (a_{11}D^2 + b_{11}D + c_{11})\, x + (a_{12}D^2 + b_{12}D + c_{12})\, y &= \phi_1(t) \\ (a_{21}D^2 + b_{21}D + c_{21})\, x + (a_{22}D^2 + b_{22}D + c_{22})\, y &= \phi_2(t) \end{aligned}$$

or, more compactly,

$$\begin{aligned} P_{11}(D)\, x + P_{12}(D)\, y &= \phi_1(t) \\ P_{21}(D)\, x + P_{22}(D)\, y &= \phi_2(t) \end{aligned}$$

The above equation can be solved as a system of simultaneous equations of x and y:

$$\begin{vmatrix} P_{11}(D) & P_{12}(D) \\ P_{21}(D) & P_{22}(D) \end{vmatrix} x = \begin{vmatrix} \phi_1(t) & P_{12}(D) \\ \phi_2(t) & P_{22}(D) \end{vmatrix}$$

and $$\begin{vmatrix} P_{11}(D) & P_{12}(D) \\ P_{21}(D) & P_{22}(D) \end{vmatrix} y = \begin{vmatrix} P_{11}(D) & \phi_1(t) \\ P_{21}(D) & \phi_2(t) \end{vmatrix}$$

or $$[P_{11}(D)\, P_{22}(D) - P_{12}(D)\, P_{21}(D)]\, x = P_{22}(D)\, \phi_1(t) - P_{12}(D)\, \phi_2(t)$$

$$[P_{11}(D)\, P_{22}(D) - P_{12}(D)\, P_{21}(D)]\, y = P_{11}(D)\, \phi_2(t) - P_{21}(D)\, \phi_1(t)$$

[Theorem]

If the determinant of the operational coefficients of a system of n linear differential equations with **constant coefficients** is **not identically zero**, then the **total number of independent arbitrary constants** in all complete solutions of the system is equal to **the degree of the determinant of the operational coefficients**, regarded as a polynomial in D.

In particular cases in which the determinant of the operational coefficients is **identically zero**, the system may have **no solution** or it may have **solutions containing any number of independent constants.**

[Example] $2x'' + 3x' - 9x + y'' + 7y' - 14y = 4$

$x' + x + y' + 2y = -8e^{2t}$

[Solution] The above equations can be written as

$$(2D^2 + 3D - 9)\,x + (D^2 + 7D - 14)\,y = 4$$
$$(D + 1)\,x + (D + 2)\,y = -8e^{2t}$$

thus

$$\begin{vmatrix} 2D^2 + 3D - 9 & D^2 + 7D - 14 \\ D + 1 & D + 2 \end{vmatrix} x = \begin{vmatrix} 4 & D^2 + 7D - 14 \\ -8e^{2t} & D + 2 \end{vmatrix}$$

or $$(D^3 - D^2 + 4D - 4)\,x = 8 + 32e^{2t}$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + 4\lambda - 4 = 0 \quad \therefore \lambda = 1, \text{ and } \lambda = \pm 2i$$

$$\therefore \quad x = c_1 \cos 2t + c_2 \sin 2t + c_3 e^t - 2 + 4e^{2t}$$

Similarly,

$$\begin{vmatrix} 2D^2 + 3D - 9 & D^2 + 7D - 14 \\ D + 1 & D + 2 \end{vmatrix} y = \begin{vmatrix} 2D^2 + 3D - 9 & 4 \\ D + 1 & -8e^{2t} \end{vmatrix}$$

or $$(D^3 - D^2 + 4D - 4)\,y = -4 - 40e^{2t}$$

$$\Rightarrow \quad y = k_1 \cos 2t + k_2 \sin 2t + k_3 e^t + 1 - 5e^{2t}$$

We now have arbitrary constants c_1, c_2, c_3, k_1, k_2, and k_3. However, the number of arbitrary constants is this system should be 3 (degree of the determinant is 3). We need to find the relationship among c_1, c_2, c_3, k_1, k_2, and k_3 in the following:

Put the above solution into any one of the original equations, say,

$$(D+1)\,x + (D+2)\,y = -8e^{2t},$$

we have

$$\begin{aligned}&(c_1 + 2c_2 + 2k_1 + 2k_2)\cos 2t \\ &\qquad + (-2c_1 + c_2 - 2k_1 + 2k_2)\sin 2t + (2c_3 + 3k_3)e^t - 8e^{2t} \\ &\qquad = -8e^{2t}\end{aligned}$$

or

$$c_1 + 2c_2 + 2k_1 + 2k_2 = 0$$

$$-2c_1 + c_2 - 2k_1 + 2k_2 = 0$$

$$2c_3 + 3k_3 = 0$$

thus

$$k_1 = \frac{-3c_1 - c_2}{4}$$

$$k_2 = \frac{c_1 - 3c_2}{4}$$

$$k_3 = -\frac{2}{3}c_3$$

$$\therefore \quad x = c_1\cos 2t + c_2\sin 2t + c_3e^t - 2 + 4e^{2t}$$

$$y = -\frac{1}{4}(3c_1 + c_2)\cos 2t + \frac{1}{4}(c_1 - 3c_2)\sin 2t - \frac{2}{3}c_3e^t + 1 - 5e^{2t}$$

[Exercise] Solve

$$(D-1)\,x - y = 0$$
$$-2x + (D-1)\,y - z = 0$$
$$-2y + (D-1)\,z = 6e^{2t}$$

[Ans:]

$$x = -(c_1/2)\,e^{t} + (c_2/2)\,e^{-t} + (c_3/2)e^{3t} - 2e^{2t}$$
$$y = -c_2\,e^{-t} + c_3\,e^{3t} - 2e^{2t}$$
$$z = c_1\,e^{t} + c_2\,e^{-t} + c_3\,e^{3t} + 2e^{2t}$$

(Check the Answer!)